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ON THE PROBABILITY OF COVERING THE CIRCLE BY RANDOM
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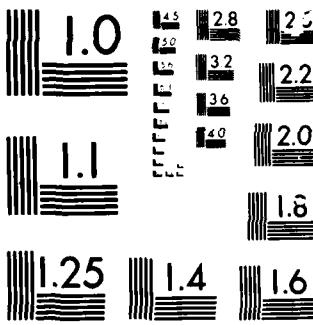
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BY

F. W. HUPFER and L. A. SHEPP

TECHNICAL REPORT NO. 372

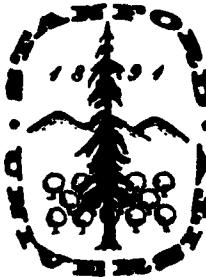
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On the Probability of Covering the
Circle by Random Arcs

By

F. W. Huffer and L. A. Shepp

1. Introduction.

Suppose arcs of length ℓ_k , $0 < \ell_k < 1$, $1 \leq k \leq n$, are thrown independently and uniformly on a circumference C having unit length. Let $P(\ell_1, \ell_2, \dots, \ell_n)$ be the probability that C is completely covered by the n random arcs. Stevens (1939) has explicitly evaluated $P(\ell_1, \ell_2, \dots, \ell_n)$ in the case $\ell_1 = \ell_2 = \dots = \ell_n$. It seems hopeless to give a simple formula for the case of general arc lengths.

There is an extensive literature concerned with random arcs and coverage problems. For a survey of some of this literature see Solomon (1978). Much of this literature deals exclusively with the case of equal arc lengths. We shall briefly mention some work dealing with coverage of the circle by arcs of differing lengths.

Consider the following situation: infinitely many arcs with lengths $\ell_1, \ell_2, \ell_3, \dots$ are placed randomly on the circumference C . A number of authors studied conditions on the sequence $\{\ell_n\}$ which ensure the complete coverage of C (with probability one). This work culminated in a necessary and sufficient condition given by Shepp (1972).

Other work has dealt with arcs having random lengths. For example, Siegel and Holst (1982) give an expression for the probability that C is completely covered by n randomly placed arcs whose lengths are i.i.d. random variables. This expression, although complicated, may be explicitly evaluated in some cases.

In this paper we examine qualitative properties of the function $P(\ell_1, \ell_2, \dots, \ell_n)$. In particular, we show that

$$(1.1) \quad P(\ell_1, \ell_2, \dots, \ell_n) \text{ is Schur-convex,}$$

and

$$(1.2) \quad P(\ell_1, \ell_2, \dots, \ell_n) \text{ is convex in each argument} \\ \text{(keeping the others fixed).}$$

As an immediate consequence of (1.1) and (1.2) we obtain the next result:

Suppose we have a collection of n arcs with lengths $\ell_1, \ell_2, \dots, \ell_n$. Randomly choose k arcs without replacement from this collection and throw these arcs independently and uniformly on C . Let $P_k(\ell_1, \ell_2, \dots, \ell_n)$ be the probability that these k arcs cover C completely. Then

$$(1.3) \quad P_k(\ell_1, \ell_2, \dots, \ell_n) \text{ is Schur-convex.}$$

These three results are proved in sections 2, 3 and 4 respectively. See Marshall and Olkin (1979) for information on Schur-convexity.

Our main result is (1.1) which was proposed as a conjecture by Frank Proschan. In addition to improving our qualitative understanding of the function $P(l_1, l_2, \dots, l_n)$, Schur-convexity allows us to compute explicit bounds in some cases, for instance,

$$P(l_1, l_2, \dots, l_n) \geq P(\bar{l}, \bar{l}, \dots, \bar{l})$$

where $\bar{l} = n^{-1} \sum_i l_i$ and $P(\bar{l}, \bar{l}, \dots, \bar{l})$ may be evaluated using Stevens' formula.

The result (1.3) is closely related to Example 3 in Section 4 of Huffer (1986). We shall briefly describe this example. Let $P(k, F)$ denote the probability that C is covered by k randomly placed arcs whose lengths are i.i.d. with distribution F . It is shown that $P(k, F) \leq P(k, G)$ whenever G is a dilation of F , that is,

$$\int_0^1 \phi(x) dF(x) \leq \int_0^1 \phi(x) dG(x)$$

for all convex functions ϕ . This dilation ordering is the analog for distributions of the majorization ordering for vectors (see page 16 of Marshall and Olkin (1979)). Thus we may view the above result as the limiting case of (1.3) as $n \rightarrow \infty$ and sampling without replacement becomes indistinguishable from sampling with replacement.

2. Proof of Schur-convexity.

Assume without loss of generality that $l_1 \geq l_2$. To demonstrate Schur-convexity it suffices to show that

$$(2.1) \quad P(l_1, l_2, l_3, \dots, l_n) \leq P(l_1 + \epsilon, l_2 - \epsilon, l_3, \dots, l_n)$$

for all sufficiently small $\epsilon > 0$.

An outline of the proof follows. To prove (2.1) we must compare two situations: one in which the first two arcs have lengths l_1, l_2 and the other in which these arcs have lengths $l_1 + \epsilon, l_2 - \epsilon$. To do this we construct these two situations on the same probability space and then condition on the positions of the first two arcs. Having fixed the first two arcs, we examine the conditional probability of covering the remaining part of the circle by the remaining $n-2$ arcs with lengths l_3, \dots, l_n . It is easy to handle the case when the first two arcs overlap. When the first two arcs are disjoint, a more complicated proof is needed. For this case we use a reflection argument to express the coverage probability inequality in the form given in (2.5). A further conditional argument then reduces this to a consequence of the lemma given at the close of this section.

For notational convenience, we parameterize the circle by using the real line modulo one. A real number x corresponds to a unique point on C which we denote by $[x]$. $[x] = [y]$ if and only if $x-y$ is an integer. An interval (a, b) on the line corresponds to a unique arc on C which we denote by $[(a, b)]$.

Let X_1, X_2, \dots, X_n be independent and uniformly distributed on $(0, 1)$. Define

$$A_1 = [(x_1, x_1 + \ell_1)], A_2 = [(x_2, x_2 + \ell_2)],$$

$$A'_1 = [(x_1, x_1 + \ell_1 + \varepsilon)], A'_2 = [(x_2 + \varepsilon, x_2 + \ell_2)]$$

and

$$H = \bigcup_{i=3}^n [(x_i, x_i + \ell_i)].$$

Formula (2.1) is equivalent to

$$(2.2) \quad P\{C \subset A_1 \cup A_2 \cup H\} \leq P\{C \subset A'_1 \cup A'_2 \cup H\}.$$

To prove (2.2) we show that

$$(2.3) \quad P\{C \subset A_1 \cup A_2 \cup H | x_1, x_2\} \leq P\{C \subset A'_1 \cup A'_2 \cup H | x_1, x_2\}$$

almost surely. There are 3 cases to consider:

(i) $[x_2] \in A_1$,

(ii) $[x_2] \notin A_1$ and $[x_2 + \ell_2] \in A_1$,

(iii) $A_1 \cap A_2 = \emptyset$.

In case (i) we have $A_1 \cup A_2 \subset A'_1 \cup A'_2$. In case (ii) both $A_1 \cup A_2$ and $A'_1 \cup A'_2$ are arcs on C and the length of $A_1 \cup A_2$ is less than or equal to the length of $A'_1 \cup A'_2$. The reader can easily verify these facts. Thus (2.3) holds in both cases (i) and (ii).

The remainder of the proof deals with case (iii) so that in the following we take x_1 and x_2 to be fixed (nonrandom) values such that $A_1 \cap A_2 = \emptyset$. This implies $A'_1 \cap A'_2 = \emptyset$. Let V and V' denote the

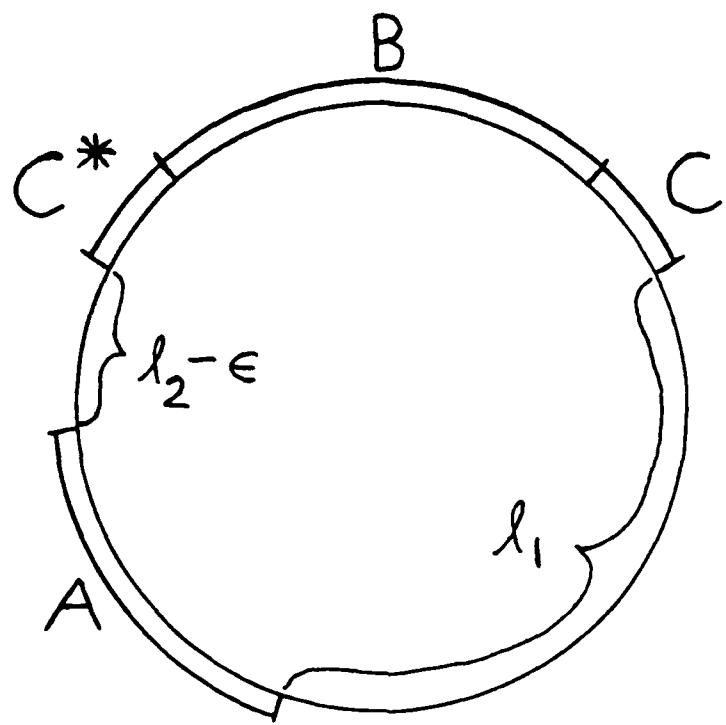


Diagram 1

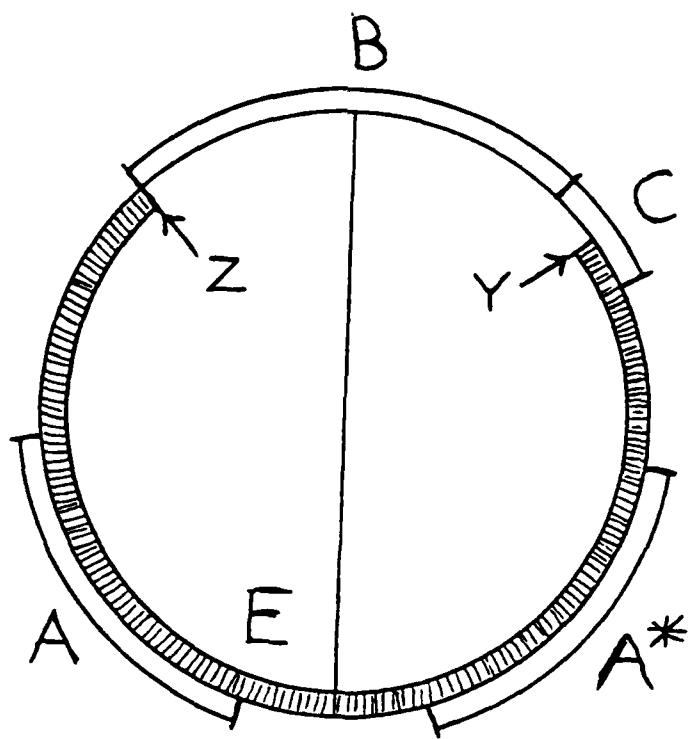


Diagram 2

complements of $A_1 \cup A_2$ and $A'_1 \cup A'_2$ respectively. We can now rewrite (2.3) as

$$(2.4) \quad P\{V \subset H\} \leq P\{V' \subset H\}.$$

V consists of two disjoint arcs separated by a distance of ℓ_2 . V' consists of disjoint arcs having the same lengths but situated closer together (separated by $\ell_2 - \varepsilon$). Let A, B, C and C^* be the disjoint arcs given in diagram 1. Both C and C^* have length ε .

With this notation $V = A \cup B \cup C$ and $V' = A \cup B \cup C^*$.

Let A^* be the reflection of A about the diagonal through the midpoint of B . Noting that the reflection of C is C^* and the reflection of B is B , symmetry yields

$$P\{A \cup B \cup C^* \subset H\} = P\{A^* \cup B \cup C \subset H\}$$

so that (2.4) is equivalent to

$$P\{A \cup B \cup C \subset H\} \leq P\{A^* \cup B \cup C \subset H\}.$$

Since

$$P\{A \cup B \cup C \subset H\} = P\{A \cup B \subset H\} - P\{A \cup B \subset H, C \notin H\}$$

and by symmetry $P\{A \cup B \subset H\} = P\{A^* \cup B \subset H\}$, this may be rewritten as

$$P\{A \cup B \subset H, C \notin H\} \geq P\{A^* \cup B \subset H, C \notin H\}$$

or equivalently

$$(2.5) \quad P\{A \subset H | B \subset H, C \notin H\} \geq P\{A^* \subset H | B \subset H, C \notin H\} .$$

In proving (2.5), in addition to conditioning on the event $\{B \subset H, C \notin H\}$, we shall assume we know which of the random arcs intersect B and the exact position of these arcs. More precisely, we are given $\{B \subset H, C \notin H\}$ and the value X_i for all i such that $[(X_i, X_i + \ell_i)]$ intersects B . For convenience we use F to denote this conditioning. To prove (2.5) and complete the proof of our theorem it suffices to show that

$$(2.6) \quad P\{A \subset H | F\} \geq P\{A^* \subset H | F\} .$$

Let D be the union of all the random arcs $[(X_i, X_i + \ell_i)]$ which intersect B . Given F , we know that D is an arc containing B and that one of its endpoints (call it Y) must lie in C . We can ignore the case where D is very large and has both endpoints in C because in that case both conditional probabilities in (2.6) are equal to one.

At this point we shall assume that ϵ (which is the length of C) is less than the minimum of $\ell_3, \ell_4, \dots, \ell_n$. Let ξ be the collection of those random arcs which do not intersect B , that is,

$$\xi = \{i | i \geq 3 \text{ and } B \cap [(X_i, X_i + \ell_i)] = \emptyset\}.$$

Given F , the arcs in ξ cannot intersect $D \cap C$, for this would lead to complete coverage of C . Thus, given F , the arcs in ξ are uniformly

and independently distributed inside the arc extending from Y to Z (where Z is the endpoint of B on the side not adjoining C) which we denote by E .

Various items introduced in the previous paragraphs are pictured in diagram 2. The diagram depicts A and A^* as being disjoint, but this will not always be the case.

Define

$$A_r = A - D, \quad A_r^* = A^* - D$$

and

$$H_r = \bigcup_{i \in \xi} [(x_i, x_i + \ell_i)] .$$

Then

$$P\{A \subset H | F\} = P\{A_r \subset H_r\}$$

and

$$P\{A^* \subset H | F\} = P\{A_r^* \subset H_r\} .$$

If D intersects A^* , then $A_r \subset A_r^*$ so that (2.6) holds. Now consider the case when D and A^* are disjoint so that $A_r^* = A^*$. Because A and A^* have the same length and A is closer than A^* to the midpoint of E , the following lemma shows that

$$P\{A \subset H_r\} \geq P\{A^* \subset H_r\} .$$

Since $A_r \subset A$ we have $P\{A_r \subset H_r\} \geq P\{A \subset H_r\}$ so that again (2.6) holds. This completes the proof.

It remains to present and prove the lemma needed above. This will be done independently of the previous material.

Lemma: Randomly place k intervals of length $\theta_1, \theta_2, \dots, \theta_k$ inside the interval $(0, L)$. The left endpoint of the interval of length θ_1 is uniformly distributed on $(0, L-\theta_1)$. Fix a value b in $(0, L)$. For $0 \leq \lambda \leq L-b$ define $X(\lambda)$ and $Q(\lambda)$ as follows. $X(\lambda)$ denotes the Lebesgue measure of the subset of $(\lambda, \lambda+b)$ which is covered by the k random intervals. $Q(\lambda)$ denotes the probability that $(\lambda, \lambda+b)$ is completely covered by the k random intervals, that is, $Q(\lambda) = P\{X(\lambda) = b\}$. For $0 \leq \lambda \leq (L-b)/2$ we have:

- (a) The distribution of $X(\lambda)$ is stochastically increasing in λ .
- (b) $Q(\lambda)$ increases with λ .

Part (b) says that the probability of complete coverage increases as the interval $(\lambda, \lambda+b)$ moves toward the center of $(0, L)$ while keeping its length fixed. Part (b) follows immediately from part (a) which is proved below.

Proof: The k random intervals will be denoted I_1, I_2, \dots, I_k . Let $B = (\lambda, \lambda+b)$ with $\lambda < (L-b)/2$. Choose ϵ sufficiently small, $\epsilon \leq (L-b-2\lambda)/2$. Instead of comparing the coverage of the intervals $B = (\lambda, \lambda+b)$ and $(\lambda+\epsilon, \lambda+\epsilon+b)$, we shall consider only B and transform the intervals I_1, I_2, \dots, I_k uniformly distributed in $L = (0, L)$ into intervals $I_1^*, I_2^*, \dots, I_k^*$ uniformly distributed in $L^* = (-\epsilon, L-\epsilon)$. An interval $I_i = (c, d)$ in L is transformed into $I_i^* = (c^*, d^*)$ in L^* according to this rule:

If $d \leq L-\epsilon$, then $(c^*, d^*) = (c, d)$.

If $d > L-\epsilon$, then $(c^*, d^*) = (d-L, d-L+(d-c))$.

Let μ denote the Lebesgue measure. The restrictions on λ and ε ensure that the midpoint of B lies to the left of the midpoint of I_i^* so that

$$(2.7) \quad \mu(B \cap I_i) \leq \mu(B \cap I_i^*) \text{ for all } i.$$

Define $\xi = \{i : I_i \subset (0, b+2\lambda)\}$, $C = B \cap (\bigcup_{i \in \xi} I_i)$, $D = B \cap (\bigcup_{i \notin \xi} I_i)$, $C^* = B \cap (\bigcup_{i \in \xi} I_i^*)$, $D^* = B \cap (\bigcup_{i \notin \xi} I_i^*)$. Part (a) may be restated as

$$(2.8) \quad \mu(C \cup D) \leq_{st} \mu(C^* \cup D^*)$$

where \leq_{st} is the stochastic ordering. Note that $C = C^*$. This follows because $b+2\lambda \leq L-\varepsilon$ so that $I_i = I_i^*$ for $i \in \xi$.

Suppose we know ξ and the exact position of I_i for all $i \notin \xi$. Conditional on this information, the sets D and D^* are fixed (nonrandom), and for $i \notin \xi$ the intervals I_i are independent and uniformly distributed inside $(0, b+2\lambda)$. Let R denote reflection about the midpoint of B . $R(C)$ and C have identical distributions because B is centered inside the interval $(0, b+2\lambda)$. Using (2.7) it is easily seen that either $D \subset R(D^*)$ or $D \subset D^*$ depending on whether or not the lefthand endpoint of D belongs to an interval I_i for which $I_i \neq I_i^*$. Therefore, either $C \cup D \subset C^* \cup R(D^*)$ or $C \cup D \subset C^* \cup D^*$. In the first case $\mu(C \cup D) \leq \mu(C^* \cup R(D^*)) = \mu(R(C^*) \cup D^*) \stackrel{d}{=} \mu(C^* \cup D^*)$ where $\stackrel{d}{=}$ denotes equality in distribution. In the second case $\mu(C \cup D) \leq \mu(C^* \cup D^*)$. This shows (2.8) holds conditionally and therefore must also hold without the conditioning. This completes the proof.

3. Convexity in each argument.

We now prove (1.2). Let X_1, X_2, \dots, X_n be independent and uniformly distributed on $(0,1)$. Choose y and z satisfying $0 < y < z < y+z < 1$. Define

$$\mathcal{B}_1 = [(X_1, X_1+z)], \mathcal{B}_2 = [(X_1+y, X_1+y+z)],$$

$$\mathcal{B}_3 = [(X_1+y, X_1+z)], \mathcal{B}_4 = [(X_1, X_1+y+z)],$$

and

$$G = \bigcup_{i=2}^n [(X_i, X_i+\ell_i)] .$$

Here we use the bracket notation described early in section 2. For $1 \leq i \leq 4$ define J_i to be the indicator of the event $\{C \in \mathcal{B}_i \cup G\}$. Because $\mathcal{B}_1 \cap \mathcal{B}_2 = \mathcal{B}_3$ and $\mathcal{B}_1 \cup \mathcal{B}_2 = \mathcal{B}_4$, it is easily seen that

$$J_1 + J_2 \leq J_3 + J_4 .$$

Taking expected values leads to

$$P(z, \ell_2, \dots, \ell_n) \leq \frac{1}{2}[P(z-y, \ell_2, \dots, \ell_n) + P(z+y, \ell_2, \dots, \ell_n)] .$$

This suffices to show that $P(z, \ell_2, \dots, \ell_n)$ is convex in z when ℓ_2, \dots, ℓ_n are held fixed. The argument above is essentially that used by Huffer (1984).

4. Proof of (1.3).

By definition we have

$$(4.1) \quad P_k(\ell_1, \ell_2, \dots, \ell_n) = \frac{k!(n-k)!}{n!} \sum P(\ell_{i_1}, \ell_{i_2}, \dots, \ell_{i_k})$$

with the sum over all k -tuples (i_1, i_2, \dots, i_k) satisfying

$1 \leq i_1 < i_2 < \dots < i_k \leq n$. Suppose that $0 < \varepsilon < \ell_2 \leq \ell_1 < \ell_1 + \varepsilon < 1$.

It suffices to show that

$$P_k(\ell_1 + \varepsilon, \ell_2 - \varepsilon, \ell_3, \dots, \ell_n) - P_k(\ell_1, \ell_2, \ell_3, \dots, \ell_n) \geq 0 .$$

Using (4.1), canceling common terms and then regrouping gives the equivalent expression

$$\begin{aligned} & \sum_x [P(\ell_1 + \varepsilon, \ell_2 - \varepsilon, x) - P(\ell_1, \ell_2, x)] \\ & + \sum_z [P(\ell_1 + \varepsilon, z) + P(\ell_2 - \varepsilon, z) - P(\ell_1, z) - P(\ell_2, z)] \geq 0 . \end{aligned}$$

The first sum is over all $(k-2)$ -tuples $x = (\ell_{i_1}, \ell_{i_2}, \dots, \ell_{i_{k-2}})$ with $3 \leq i_1 < i_2 < \dots < i_{k-2} \leq n$. The second sum is over all $(k-1)$ -tuples $z = (\ell_{j_1}, \ell_{j_2}, \dots, \ell_{j_{k-1}})$ with $3 \leq j_1 < j_2 < \dots < j_{k-1} \leq n$. Each term in the first sum is nonnegative by (1.1) and each term in the second sum is nonnegative by (1.2). This completes the proof.

Acknowledgements

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